Argon Thermometer.

Temperature. ° C.	Pressure. mm.	Volume.	R.
100·1	1414·9	1·0026	3·8095
0·0	1040·0	1·0000	3·8022
– 182·7	353·2	0·9953	3·8930

No correction has been made for the unheated or uncooled stem of the thermometer; but it is obvious that although the lowest temperature lies close to the boiling point of argon, the ratio of the values of PV/T of hydrogen and argon at that temperature, as well as the others, is practically constant.

"On some Expressions for the Radial and Axial Components of the Magnetic Force in the Interior of Solenoids of Circular Cross-section." By C. Coleridge Farr, B.Sc., formerly Angas Scholar, University of Adelaide. Communicated by Professor H. Lamb, F.R.S. Received June 7,—Read June 16, 1898.

In the present paper, certain expressions are arrived at, in terms of zonal spherical harmonics and their first derivatives, by which the values of the two components of the magnetic force may be calculated for any point in the interior of a coil, and hence the total force may be found both in magnitude and direction. The resulting series suffer from the well-known defect in the spherical harmonic method, in that they are not very rapidly converging for points near the boundary of the space for which they apply. A table of the values of the first derivatives of the first seven zonal harmonics is added. This table, in conjunction with that calculated by Messrs. Holland, Jones, and Lamb, and published in the 'Philosophical Magazine,' Series 5, December, 1891, will facilitate the numerical use of the expressions arrived at.

Let Ωda be the magnetic potential at any point within a solenoid whose depth of winding da is indefinitely small. If a be the radius of this "solenoidal sheet," and the axis of z be the axis of the coil, the axis of x being along a radius of the circular section of the coil, and the origin at the centre of the equatorial section of the coil, we have, neglecting the insulating covering of the wire,

$$X = -\int_T^S \frac{d}{dx}(\Omega da), \qquad Z = -\int_T^S \frac{d}{dy}(\Omega da),$$

where X = the radial component of the magnetic force,

Z = the axial component,

S = radius of the outside winding of the coil of finite dimensions,

T = radius of the inside winding.

Now $$\Omega da = n\gamma da\,[-4\pi z + V_1 - V_2]^* \quad \text{.............. (1)},$$

where n = the number of turns per unit length *per unit increase in radius*,

γ = current in C.G.S. units,

V_1 = potential at the point considered due to a plane area of surface density unity, coinciding with the positive end of the coil,

V_2 = potential due to the negative end.

If r and r' be the distance of the point considered from the centres of the positive and negative ends of the coil respectively, and P_n and Q_n the nth zonal harmonics corresponding to the angles θ and ϕ which r and r' make respectively with the axis of the coil, we have

$$V_1 = 2\pi\left(-rP_1 + a + \Sigma_{p=1}^{p=\infty}(-1)^{p+1}\frac{(2p)!}{(2p-1)\,2^{2p}\,(p!)^2}\,\frac{r^{2p}P_{2p}}{a^{2p-1}}\right), \dagger$$

when $r < a$, and

$$V_1 = 2\pi\left(\Sigma_{p=1}^{p=\infty}(-1)^{p+1}\frac{(2p)!}{(2p-1)\,2^{2p}\,(p!)^2}\,\frac{a^{2p}P_{2p-2}}{r^{2p-1}}\right), \dagger$$

when $r > a$, with similar expressions for V_2 in terms of r' and Q.

The following relations are true:—

$$(1)\quad \frac{d}{dx}\frac{P_\sigma}{r^{\sigma+1}} = \frac{1}{r^{\sigma+2}}\frac{d}{d\theta}P_{\sigma+1};$$

$$(2)\quad \frac{d}{dx}r^\sigma P_\sigma = r^{\sigma-1}\frac{d}{d\theta}P_{\sigma-1};$$

$$(3)\quad \frac{d}{dz}\frac{P_\sigma}{r^{\sigma+1}} = (\sigma+1)\frac{P_{\sigma+1}}{r^{\sigma+2}}; \ddagger$$

$$(4)\quad \frac{d}{dz}r^\sigma P^\sigma = -\sigma r^{\sigma-1}P_{\sigma-1}. \ddagger$$

When Q and r' are substituted for P and r, the forms of (1) and (2) remain unaltered; whilst (3) and (4) become respectively

$$(5)\quad \frac{d}{dz}\frac{Q_\sigma}{r'^{\sigma+1}} = -(\sigma+1)\frac{Q_{\sigma+1}}{r'^{\sigma+2}};$$

$$(6)\quad \frac{d}{dz}r'^\sigma Q_\sigma = \sigma r'^{\sigma-1}Q_{\sigma-1}.$$

* Maxwell's 'Electricity and Magnetism,' Second Edition, vol. 2, § 676, eq. 12.

† *Ibid.*, eqs. 14 and 15.

‡ The formulæ (3) and (4) may be generalised as follows:—

$$\frac{d^i}{dz^i}\frac{P_\sigma}{r^{\sigma+1}} = \frac{(\sigma+i)!}{\sigma!}\,\frac{P_{\sigma+i}}{r^{\sigma+i+1}},$$

$$\frac{d^i}{dz^i}r^\sigma P_\sigma = (-1)^i\frac{\sigma!}{(\sigma-i)!}r^{\sigma-i}P_{\sigma-i}.$$

These relations may all be proved (as was done originally by the author) by induction, using the well known relation

$$P_\sigma = \frac{2\sigma-1}{\sigma}\cos\theta\, P_{\sigma-1} - \frac{\sigma-1}{\sigma} P_{\sigma-2}.$$

Assuming them to be true for $P_{\sigma-1}$ and $P_{\sigma-2}$, they may then be shown to be true for P_σ; and trial establishes the equality in the case of P_2 and P_1. Professor T. R. Lyle has, however, given me a much shorter and neater proof by means of the relations

$$\begin{aligned}
&(a)\quad (1-\mu^2)\,P'_\sigma = \sigma P_{\sigma-1} - \sigma\mu P_\sigma,\\
&(b)\quad (\sigma+1)\,P_{\sigma+1} - (2\sigma+1)\,\mu P_\sigma + \sigma P_{\sigma-1} = 0,\\
&(c)\quad \mu P'_\sigma = \sigma P_\sigma + P'_{\sigma-1},\\
&(d)\quad P'_{\sigma+1} - P'_{\sigma-1} = (2\sigma+1)\,P_\sigma,
\end{aligned}$$

where, as usual, $\quad P'_\sigma = \dfrac{dP_\sigma}{d\mu}, \qquad \mu = \cos\theta = \dfrac{l-z}{r},$

if l denote the half-length of the coil. Since

$$r = \sqrt{(l-z)^2 + x^2},$$

we have $\quad \dfrac{dr}{dz} = -\mu, \qquad \dfrac{dr}{dx} = \sqrt{1-\mu^2},$

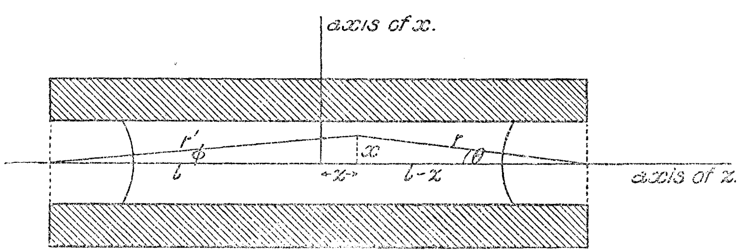
$$\frac{d\mu}{dz} = \frac{-(1-\mu^2)}{r}, \qquad \frac{d\mu}{dx} = \frac{-\mu\sqrt{1-\mu^2}}{r}.$$

A single example is sufficient. Taking the case of equation (3) we have

$$\begin{aligned}
\frac{d}{dz}\frac{P_\sigma}{r^{\sigma+1}} &= \frac{1}{r^{\sigma+2}}\left[(\sigma+1)\,\mu P_\sigma - (1-\mu^2)\,P'_\sigma\right]\\
&= (\sigma+1)\,\frac{P_{\sigma+1}}{r^{\sigma+2}}, \text{ by } (a) \text{ and } (b).
\end{aligned}$$

Case I.

Expressions for the components of the magnetic force in that region of a long coil for which r and r' are both greater than S.



It will be found that

(α) X =

$$2\pi\gamma n\left[\Sigma_{p=1}^{p=\infty}(-1)^p\frac{(2p)!\,(S^{2p+1}-T^{2p+1})}{(2p+1)(2p-1)\,2^{2p}(p!)^2}\left(\frac{1}{r^{2p}}\frac{dP_{2p-1}}{d\theta}-\frac{1}{r'^{2p}}\frac{dQ_{2p-1}}{d\theta}\right)\right];$$

$$(\beta)\quad Z = 2\pi\gamma n\left[2(S-T)\right.$$
$$\left.+\Sigma_{p=1}^{p=\infty}(-1)^p\frac{(2p)!\,(S^{2p+1}-T^{2p+1})}{(2p+1)\,2^{2p}(p!)^2}\left(\frac{P_{2p-1}}{r^{2p}}+\frac{Q_{2p-1}}{r'^{2p}}\right)\right].$$

Since $n(S-T)$ = the number of turns per unit of length, = N (say), the defect of Z from $4\pi N\gamma$ depends on the importance of the terms under the sign Σ. For the region near the equatorial plane of an infinitely long solenoid these terms become vanishingly small.

The series under the sign Σ are convergent, though not very rapidly so if S is nearly as large as r or r'.

Case II.

Expressions for the region near one end of a long solenoid where $r < T$ and $r' > S$.

In this case

$$(\gamma)\quad X = 2\pi\gamma n\left[\frac{r}{2}\frac{dP_1}{d\theta}\log_e\frac{T}{S}\right.$$
$$+\Sigma_{p=1}^{p=\infty}(-1)^p\frac{(2p+2)!\left(\frac{1}{S^{2p}}-\frac{1}{T^{2p}}\right)}{2p(2p+1)2^{2p+2}\{(p+1)!\}^2}r^{2p+1}\frac{dP_{2p+1}}{d\theta}$$
$$\left.+\Sigma_{p=1}^{p=\infty}(-1)^{p+1}\frac{(2p)!\,(S^{2p+1}-T^{2p+1})}{(2p+1)(2p-1)\,2^{2p}(p!)^2}\frac{1}{r'^{2p}}\frac{dQ_{2p-1}}{d\phi}\right];$$

$$(\delta)\quad Z = 2\pi\gamma n\left[S-T+rP_1\log_e\frac{S}{T}\right.$$
$$+\Sigma_{p=1}^{p=\infty}(-1)^{p+1}\frac{(2p)!\left(\frac{1}{S^{2p}}-\frac{1}{T^{2p}}\right)}{(2p)\,2^{2p}(p!)^2}r^{2p+1}P_{2p+1}$$
$$\left.+\Sigma_{p=1}^{p=\infty}(-1)^p\frac{(2p)!\,(S^{2p+1}-T^{2p+1})}{(2p+1)\,2^{2p}(p!)^2}\frac{Q_{2p-1}}{r'^{2p}}\right].$$

From (δ) it is easy to deduce the fact that in the plane of one end of an infinitely long solenoid the axial component is constant and equal to $2\pi N\gamma$, *i.e.*, half its value for the equatorial region of the same coil.

Case III.

Short solenoid: expressions for the components in the region in which r and r' are both less than T.
Here

$$(\epsilon)\quad X =$$

$$2\pi\gamma n\left[\Sigma_{p=1}^{p=\infty}(-1)^p\frac{(2p)!\left(\frac{1}{S^{2p}}-\frac{1}{T^{2p}}\right)}{2p\,.\,2^{2p+1}\,.\,p!\,(p+1)!}\left(r^{2p+1}\frac{dP_{2p+1}}{d\theta}-r'^{2p+1}\frac{dQ_{2p+1}}{d\phi}\right)\right];$$

$$(\zeta)\quad Z = 2\pi\gamma n\left[2l\log_e\frac{S}{T}\right.$$

$$\left.+\Sigma_{p=1}^{p=\infty}(-1)^{p+1}\frac{(2p)!\left(\frac{1}{S^{2p}}-\frac{1}{T^{2p}}\right)}{(2p)\,2^{2p}\,(p!)^2}(r^{2p+1}P_{2p+1}+r'^{2p+1}Q_{2p+1})\right].$$

It is easily deduced from (ζ) that the value of Z at the centre of a circular coil consisting of a few turns all having approximately the same radius T, is $2\pi\gamma K/T$, where K is the total number of turns.

Case IV.

Expression for the magnetic force in that region of a solenoid in which $S > r > T$ and $r' > S$.

If $$\frac{T}{r} = \epsilon < 1; \qquad \frac{r}{S} = \rho < 1,$$

we have

$$X = 2\pi\gamma n\left[\frac{r}{2}\frac{dP}{d\theta}\log_e\rho + r\,\Sigma_{p=1}^{p=\infty}(-1)^{p+1}\frac{(2p)!\,(1-\rho^{2p})}{2p.2^{2p+1}\,(p+1)!\,p!}\frac{dP_{2p+1}}{d\theta}\right.$$

$$+r\,\Sigma_{p=1}^{p=\infty}(-1)^p\frac{(2p)!\,(1-\epsilon^{2p+1})}{(2p+1)\,(2p-1)\,2^{2p}\,(p!)^2}\frac{dP_{2p-1}}{d\theta}$$

$$\left.+\Sigma_{p=1}^{p=\infty}(-1)^{p+1}\frac{(2p)!\,(S^{2p+1}-T^{2p+1})}{(2p+1)\,(2p-1)2^{2p}(p!)^2}\frac{1}{r'^{2p}}\frac{dQ_{2p-1}}{d\phi}\right];$$

$$Z = 2\pi\gamma n\left[S+r-2T-rP_1\log_e\rho\right.$$

$$+r\,\Sigma_{p=1}^{p=\infty}(-1)^p\frac{(2p)!\,(1-\rho^{2p})}{(2p)\,2^{2p}\,(p!)^2}P_{2p+1}$$

$$+r\,\Sigma_{p=1}^{p=\infty}(-1)^p\frac{(2p)!\,(1-\epsilon^{2p+1})}{(2p+1)\,2^{2p}\,(p!)^2}P_{2p-1}$$

$$\left.+\Sigma_{p=1}^{p=\infty}(-1)^p\frac{(2p)!\,(S^{2p+1}-T^{2p+1})}{(2p+1)\,2^{2p}\,(p!)^2}\frac{Q_{2p-1}}{r'^{2p}}\right].$$

The cases for which

$$r' < T \quad \text{and} \quad S > r > T,$$

$$\text{and } S > r > T \quad \text{and} \quad S > r' > T,$$

respectively, do not need separate investigation. The results, though long and cumbrous, can easily be written down from the foregoing.

The accompanying tables were originally calculated by means of the formulæ:

$$\frac{dP_0}{d\theta} = 0,$$

$$\frac{dP_1}{d\theta} = -\sin\theta,$$

$$\frac{dP_2}{d\theta} = -\tfrac{3}{2}\sin 2\theta,$$

$$\frac{dP_3}{d\theta} = -6\sin\theta + \tfrac{15}{2}\sin^3\theta,$$

$$\frac{dP_4}{d\theta} = -5\sin 2\theta + \tfrac{35}{4}\sin^2\theta\sin 2\theta,$$

$$\frac{dP_5}{d\theta} = -15\sin\theta + \tfrac{105}{2}\sin^3\theta - \tfrac{315}{8}\sin^5\theta,$$

$$\frac{dP_6}{d\theta} = -\tfrac{21}{2}\sin 2\theta + \tfrac{189}{4}\sin^2\theta\sin 2\theta - \tfrac{693}{16}\sin^4\theta\sin 2\theta,$$

$$\frac{dP_7}{d\theta} = -28\sin\theta + 189\sin^3\theta - \tfrac{693}{2}\sin^5\theta + \tfrac{3003}{16}\sin^7\theta.$$

The results were finally checked by means of the relation

$$\frac{dP_{n+1}}{d\theta} = \frac{dP_{n-1}}{d\theta} - (2n+1)P_n\sin\theta,$$

using for this purpose the table of P_n calculated by Messrs. Holland, Jones, and Lamb, and published by Professor Perry in the 'Phil. Mag.,' vol. 32, 1891. This is not a complete check on the fourth figure in my tables, but no disagreement larger than 0·0004 has been passed over without examination, and that only in the higher values of $dP_6/d\theta$. It is therefore hoped that the tables may be found accurate as a whole, though it is perhaps scarcely to be expected but that some errors have escaped detection.

With the aid of Perry's tables and those now given the magnetic force at any point inside a coil may be found numerically by means of the formulæ in the body of the paper. For points outside the coil a

slight alteration of the expressions for Z is necessary; those for X remain the same.

It has been thought sufficient for numerical application to give $dP_7/d\theta$ to three places of decimals. The establishment of the fourth place would involve a considerable amount of extra labour. Though $dP_2/d\theta$, $dP_4/d\theta$, and $dP_6/d\theta$ are not used in the present work, they have been calculated in order to complete the table as far as it goes.

Numerical Values of the Derivatives of the First Seven Zonal Harmonics, at Intervals of One Degree.

$\frac{2\theta}{\pi}$.	$\frac{dP_1}{d\theta}$.	$\frac{dP_2}{d\theta}$.	$\frac{dP_3}{d\theta}$.	$\frac{dP_4}{d\theta}$.	$\frac{dP_5}{d\theta}$.	$\frac{dP_6}{d\theta}$.	$\frac{dP_7}{d\theta}$.	$\frac{2\theta}{\pi}$.
0	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000	0·000	0
1	−0·0175	−0·0523	−0·1047	−0·1744	−0·2615	−0·3659	−0·488	1
2	−0·0349	−0·1046	−0·2091	−0·3480	−0·5213	−0·7284	−0·969	2
3	−0·0523	−0·1568	−0·3129	−0·5201	−0·7775	−1·0841	−1·438	3
4	−0·0698	−0·2088	−0·4160	−0·6901	−1·0286	−1·4295	−1·890	4
5	−0·0872	−0·2605	−0·5180	−0·8570	−1·2728	−1·7614	−2·317	5
6	−0·1045	−0·3119	−0·6186	−1·0197	−1·5085	−2·0768	−2·715	6
7	−0·1219	−0·3629	−0·7176	−1·1781	−1·7341	−2·3727	−3·030	7
8	−0·1392	−0·4135	−0·8148	−1·3315	−1·9481	−2·6465	−3·405	8
9	−0·1564	−0·4635	−0·9099	−1·4789	−2·1492	−2·8954	−3·689	9
10	−0·1736	−0·5130	−1·0026	−1·6199	−2·3360	−3·1174	−3·926	10
11	−0·1908	−0·5619	−1·0928	−1·7537	−2·5074	−3·3104	−4·116	11
12	−0·2079	−0·6101	−1·1801	−1·8799	−2·6621	−3·4729	−4·254	12
13	−0·2250	−0·6576	−1·2643	−1·9978	−2·7993	−3·6034	−4·341	13
14	−0·2419	−0·7042	−1·3453	−2·1069	−2·9181	−3·7008	−4·376	14
15	−0·2588	−0·7500	−1·4229	−2·2069	−3·0178	−3·7646	−4·358	15
16	−0·2756	−0·7949	−1·4968	−2·2973	−3·0978	−3·7943	−4·288	16
17	−0·2924	−0·8388	−1·5668	−2·3777	−3·1576	−3·7899	−4·169	17
18	−0·3090	−0·8817	−1·6328	−2·4478	−3·1970	−3·7518	−4·001	18
19	−0·3256	−0·9235	−1·6946	−2·5073	−3·2158	−3·6806	−3·788	19
20	−0·3420	−0·9642	−1·7520	−2·5560	−3·2141	−3·5774	−3·524	20
21	−0·3584	−1·0037	−1·8050	−2·5937	−3·1920	−3·4435	−3·241	21
22	−0·3746	−1·0420	−1·8534	−2·6203	−3·1497	−3·2804	−2·915	22
23	−0·3907	−1·0790	−1·8970	−2·6357	−3·0877	−3·0917	−2·561	23
24	−0·4067	−1·1147	−1·9357	−2·6400	−3·0067	−2·8749	−2·183	24
25	−0·4226	−1·1491	−1·9696	−2·6330	−2·9073	−2·6371	−1·787	25

Numerical Values of the Derivatives of the First Seven Zonal Harmonics, at Intervals of One Degree—*continued.*

$\frac{2\theta}{\pi}$.	$\frac{dP_1}{d\theta}$.	$\frac{dP_2}{d\theta}$.	$\frac{dP_3}{d\theta}$.	$\frac{dP_4}{d\theta}$.	$\frac{dP_5}{d\theta}$.	$\frac{dP_6}{d\theta}$.	$\frac{dP_7}{d\theta}$.	$\frac{2\theta}{\pi}$.
26	−0·4384	−1·1820	−1·9984	−2·6150	−2·7903	−2·3794	−1·378	26
27	−0·4540	−1·2135	−2·0222	−2·5861	−2·6568	−2·1046	−0·963	27
28	−0·4695	−1·2436	−2·0408	−2·5464	−2·5077	−1·8156	−0·548	28
29	−0·4848	−1·2721	−2·0542	−2·4961	−2·3443	−1·5155	−0·136	29
30	−0·5000	−1·2990	−2·0625	−2·4357	−2·1680	−1·2077	+0·263	30
31	−0·5150	−1·3244	−2·0654	−2·3654	−1·9799	−0·8953	+0·647	31
32	−0·5299	−1·3482	−2·0635	−2·2855	−1·7817	−0·5815	+1·010	32
33	−0·5446	−1·3703	−2·0562	−2·1966	−1·5748	−0·2697	+1·347	33
34	−0·5592	−1·3908	−2·0437	−2·0991	−1·3608	+0·0370	+1·654	34
35	−0·5736	−1·4095	−2·0262	−1·9934	−1·1413	+0·3354	+1·927	35
36	−0·5878	−1·4266	−2·0036	−1·8802	−0·9179	+0·6225	+2·162	36
37	−0·6018	−1·4419	−1·9761	−1·7600	−0·6924	+0·8955	+2·357	37
38	−0·6157	−1·4554	−1·9438	−1·6334	−0·4664	+1·1516	+2·510	38
39	−0·6293	−1·4672	−1·9066	−1·5011	−0·2415	+1·3885	+2·620	39
40	−0·6428	−1·4772	−1·8648	−1·3637	−0·0194	+1·6038	+2·684	40
41	−0·6561	−1·4854	−1·8185	−1·2219	+0·1983	+1·7955	+2·705	41
42	−0·6691	−1·4918	−1·7678	−1·0764	+0·4100	+1·9620	+2·681	42
43	−0·6820	−1·4963	−1·7129	−0·9279	+0·6142	+2·1017	+2·614	43
44	−0·6947	−1·4991	−1·6539	−0·7772	+0·8095	+2·2136	+2·506	44
45	−0·7071	−1·5000	−1·5910	−0·6250	+0·9943	+2·2969	+2·359	45
46	−0·7193	−1·4991	−1·5244	−0·4720	+1·1677	+2·3510	+2·176	46
47	−0·7314	−1·4963	−1·4542	−0·3190	+1·3282	+2·3757	+1·961	47
48	−0·7431	−1·4918	−1·3808	−0·1668	+1·4749	+2·3715	+1·717	48
49	−0·7547	−1·4854	−1·3042	−0·0160	+1·6067	+2·3382	+1·449	49
50	−0·7660	−1·4772	−1·2248	+0·1327	+1·7229	+2·2772	+1·161	50

$\frac{2\theta}{\pi}$.	$\frac{dP_1}{d\theta}$.	$\frac{dP_2}{d\theta}$.	$\frac{dP_3}{d\theta}$.	$\frac{dP_4}{d\theta}$.	$\frac{dP_5}{d\theta}$.	$\frac{dP_6}{d\theta}$.	$\frac{dP_7}{d\theta}$.	$\frac{2\theta}{\pi}$.
51	−0·7771	−1·4672	−1·1427	+0·2784	+1·8225	+2·1892	+0·859	51
52	−0·7880	−1·4554	−1·0581	+0·4205	+1·9053	+2·0760	+0·546	52
53	−0·7986	−1·4419	−0·9714	+0·5584	+1·9704	+1·9387	+0·230	53
54	−0·8090	−1·4266	−0·8828	+0·6914	+2·0173	+1·7797	−0·088	54
55	−0·8191	−1·4095	−0·7925	+0·8188	+2·0474	+1·6009	−0·399	55
56	−0·8290	−1·3908	−0·7007	+0·9401	+2·0587	+1·4046	−0·700	56
57	−0·8387	−1·3703	−0·6078	+1·0546	+2·0521	+1·1936	−0·986	57
58	−0·8480	−1·3482	−0·5140	+1·1620	+2·0281	+0·9699	−1·252	58
59	−0·8572	−1·3244	−0·4196	+1·2617	+1·9866	+0·7369	−1·495	59
60	−0·8660	−1·2990	−0·3248	+1·3532	+1·9283	+0·4973	−1·711	60
61	−0·8746	−1·2721	−0·2299	+1·4361	+1·8538	+0·2540	−1·896	61
62	−0·8829	−1·2436	−0·1351	+1·5101	+1·7640	+0·0099	−2·048	62
63	−0·8910	−1·2135	−0·0409	+1·5748	+1·6596	−0·2321	−2·165	63
64	−0·8988	−1·1820	+0·0527	+1·6306	+1·5420	−0·4691	−2·244	64
65	−0·9063	−1·1491	+0·1454	+1·6755	+1·4114	−0·6983	−2·286	65
66	−0·9135	−1·1147	+0·2368	+1·7111	+1·2698	−0·9170	−2·290	66
67	−0·9205	−1·0790	+0·3268	+1·7366	+1·1187	−1·1226	−2·255	67
68	−0·9272	−1·0420	+0·4149	+1·7520	+0·9580	−1·3129	−2·183	68
69	−0·9336	−1·0037	+0·5011	+1·7574	+0·7906	−1·4855	−2·076	69
70	−0·9397	−0·9642	+0·5851	+1·7525	+0·6173	−1·6386	−1·984	70
71	−0·9455	−0·9235	+0·6666	+1·7376	+0·4397	−1·7704	−1·762	71
72	−0·9511	−0·8817	+0·7455	+1·7131	+0·2594	−1·8794	−1·561	72
73	−0·9563	−0·8388	+0·8214	+1·6787	+0·0776	−1·9646	−1·334	73
74	−0·9613	−0·7949	+0·8941	+1·6349	−0·1037	−2·0248	−1·088	74
75	−0·9659	−0·7500	+0·9636	+1·5819	−0·2833	−2·0596	−0·824	75

Numerical Values of the Derivatives of the First Seven Zonal Harmonics, at Intervals of One Degree—*continued.*

$\frac{2\theta}{\pi}$.	$\frac{dP_1}{d\theta}$.	$\frac{dP_2}{d\theta}$.	$\frac{dP_3}{d\theta}$.	$\frac{dP_4}{d\theta}$.	$\frac{dP_5}{d\theta}$.	$\frac{dP_6}{d\theta}$.	$\frac{dP_7}{d\theta}$.	$\frac{2\theta}{\pi}$.
76	−0·9703	−0·7042	+1·0295	+1·5200	−0·4595	−2·0687	−0·548	76
77	−0·9744	−0·6576	+1·0918	+1·4498	−0·6309	−2·0520	−0·264	77
78	−0·9781	−0·6101	+1·1501	+1·3714	−0·7961	−2·0098	+0·023	78
79	−0·9816	−0·5619	+1·2044	+1·2854	−0·9537	−1·9428	+0·309	79
80	−0·9848	−0·5130	+1·2545	+1·1923	−1·1023	−1·8519	+0·589	80
81	−0·9877	−0·4635	+1·3003	+1·0926	−1·2407	−1·7382	+0·858	81
82	−0·9903	−0·4135	+1·3416	+0·9870	−1·3679	−1·6032	+1·112	82
83	−0·9925	−0·3629	+1·3783	+0·8758	−1·4831	−1·4487	+1·348	83
84	−0·9945	−0·3119	+1·4103	+0·7598	−1·5841	−1·2760	+1·559	84
85	−0·9962	−0·2605	+1·4376	+0·6396	−1·6715	−1·0881	+1·745	85
86	−0·9976	−0·2088	+1·4599	+0·5160	−1·7440	−0·8868	+1·901	86
87	−0·9986	−0·1568	+1·4774	+0·3895	−1·8009	−0·6747	+2·024	87
88	−0·9994	−0·1046	+1·4899	+0·2608	−1·8420	−0·4544	+2·115	88
89	−0·9998	−0·0523	+1·4975	+0·1308	−1·8667	−0·2286	+2·169	89
90	−1·0000	0·0000	+1·5000	0·0000	−1·8750	0·0000	+2·187	90